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A RIGID ELLIPTIC INCLUSION IN AN ANISOTROPIC ELASTIC WHOLE SPACE

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Abstract—We consider a rigid elliptic disc embedded in a homogeneous anisotropic elastic whole space and bonded to it. The elastic field is determined corresponding to an arbitrary displacement of the disc, particular attention being devoted to the stress distribution on either side of the disc. The problem of a uniform stress field perturbed by the disc is also considered.

1. INTRODUCTION

We consider a rigid elliptic disc embedded in a whole space of homogeneous generally anisotropic elastic material and bonded to it. The greater part of this paper concerns the determination of the elastic field induced in the whole space when the disc is given an arbitrary displacement (translation plus rotation), but consideration is also given to the problem of the disc perturbing a uniform stress field, this problem being somewhat analogous to that of the rotation of the disc.

The foregoing problems have been dealt with for the case of isotropic media, and we now give some references to work in this area. Kasser and Sih [1] have considered the problem of translation of the disc and the problem of perturbation of a uniform stress field by the disc. Lur'e [2] has examined the problem of an ellipsoidal inclusion undergoing translation and rotation. The work of Eshelby [3, 4] deals *inter alia* with the perturbation of a uniform stress field by an ellipsoidal inclusion.

The methods used in this paper are based on the work of Willis [5, 6] in somewhat analogous contexts. In [5] he considers the problem of a rigid punch of elliptic cross-section indenting a homogeneous generally anisotropic half space; he established a theorem therein which overcomes the apparent difficulty of the unavailability, in closed form, of the solution to the fundamental or point force problem. This latter theorem again forms the basis for the solution of problems concerning an elliptic crack in a homogeneous generally anisotropic elastic whole space [6].

The plan of the paper is as follows. In Sections 2–5 the problem concerning the displacement of the disc is dealt with, while Section 6 deals with the problem of the perturbation by the disc of a uniform stress field. In Section 2 the displacement boundary value problem is formulated in the former case, while Section 3 outlines the basis for its solution. To begin with, the displacement field is represented by a superposition or a convolution integral involving an unknown body force layer distribution coinciding with the disc, together with the Green's functions corresponding to displacements due to point forces in the full space without inclusion. Guided by previous work, a body force layer distribution is postulated which involves nine undetermined constants. It is then shown that the displacement field thus obtained reduces to the required form on the disc provided that the nine constants are chosen suitably; they involve line integrals of quantities dependent on (readily obtainable) Fourier transforms of the Green's functions. The theorem of Willis [5] enables us to overcome the unavailability, in general, of the Green's functions in closed form. In Section 4 a detailed examination is made of the special case when the elliptic disc undergoes a translation only. The displacements are given in terms of reasonably tractable line integrals involving the Fourier transforms of the Green's functions, and the resultant forces required to translate the disc are obtained as integrals of a similar type. The stresses on either side of the disc are investigated and are shown to be remarkable in their universality: suppose that the elliptic disc is given by

$$1 - x_1^2/a_1^2 - x_2^2/a_2^2 > 0, \quad x_3 = 0,$$

with respect to rectangular cartesian coordinates Ox_i, and suppose that P_i are the corresponding components of the resultant forces required to translate the disc, then the cartesian stress components on either side of the disc $x_3 = 0\pm$ are given by

$$\tau_{3i} = \mp P_i / \{4\pi a_1 a_2 (1 - x_1^2/a_1^2 - x_2^2/a_2^2)^{1/2}\}$$

respectively. Likewise, in Section 4 detailed examination is made of the complementary case where the disc undergoes rotation about its centre. Again the displacements are given in terms of reasonably tractable integrals involving Fourier transforms of the Green's function, as are also the torques necessary to accomplish the rotations. The stresses on either side of the disc are investigated, and whereas it is not possible to express them in universal terms, analogous to the case of translation, a certain amount of progress in this direction is possible if the existence of planes of elastic symmetry is assumed.

2. FORMULATION OF THE PROBLEM

Let (x_1, x_2, x_3) be rectangular cartesian co-ordinates. A rigid elliptic disc defined by

$$x_1^2/a_1^2 + x_2^2/a_2^2 = 1, \quad x_3 = 0, \tag{2.1}$$

is embedded in a full space of generally homogeneous anisotropic elastic material and is bonded to it. As usual, τ_{ij}, u_i denote the (cartesian) stress components and displacement components respectively, the two being related by

$$\tau_{ij} = c_{ijkl} \partial u_k / \partial x_l \tag{2.2}$$

in which the constants c_{ijkl} satisfy

$$c_{ijkl} = c_{jikl}, \quad c_{ijkl} = c_{ijlk}, \quad c_{ijkl} = c_{klij}. \tag{2.3}$$

The summation convention applies throughout, Latin and Greek indices taking the values 1, 2, 3 and 1, 2 respectively, unless otherwise stated or implied.

We consider the problem of determining the elastic field induced by a general movement of the disc—a displacement α_i and a rotation ω_i about the origin, say. Mathematically, the displacement boundary value problem is that of solving

$$c_{ijkl} \partial^2 u_k / \partial x_j \partial x_l = 0, \tag{2.4}$$

subject to the boundary consitions

$$u_i = \alpha_i + e_{ijk} \omega_j x_k \quad \text{on} \quad \{x_i : 1 - x_1^2/a_1^2 - x_2^2/a_2^2 > 0, \quad x_3 = 0\}$$

and

$$u_i \to 0 \quad \text{as} \quad x_i x_i \to \infty, \tag{2.5}$$

e_{ijk} being the usual permutation symbol.

3. BASIS FOR SOLUTION

Let $v_{si}(x_1, x_2, x_3)$ be the ith component of displacement in the elastic whole space (without inclusion) due to a unit point force acting at the origin in the sth direction and vanishing at infinity. We have

$$\begin{gathered} c_{ijkl} \partial^2 v_{sk} / \partial x_j \partial x_l + \delta_{si} \delta(x_1) \delta(x_2) \delta(x_3) = 0, \\ v_{sk} \to 0 \quad \text{as} \quad x_i x_i \to \infty, \end{gathered} \tag{3.1}$$

$\delta(x_1)$ etc. denoting Dirac deltas, δ_{si} being a Kronecker delta. We note that, as a consequence of $(2.3)_3$, $v_{si} = v_{is}$. The displacement field u_i, vanishing at infinity, which is induced in the elastic whole space (without inclusion) by a body force field $F_i(x_1, x_2, x_3)$ confined to a finite part of the space (say) is given by the superposition

$$u_i = \int\int\int F_s(x_1', x_2', x_3')v_{si}(x_1 - x_1', x_2 - x_2', x_3 - x_3')\,dx_1'\,dx_2'\,dx_3'. \quad (3.2)$$

Our problem is partially solved if we can determine a suitable body force field F_s to simulate the effect of the displaced disc. The solution to the problem in the isotropic case and the work of Willis [5] suggest the choice

$$F_s = (d_s + d_{s\alpha}x_\alpha)(1 - x_1^2/a_1^2 - x_2^2/a_2^2)^{-1/2}\delta(x_3)H(1 - x_1^2/a_1^2 - x_2^2/a_2^2) \quad (3.3)$$

where H denotes the Heaviside unit function, and where d_s, $d_{s\alpha}$ are constants to be determined. Using (3.2), the displacement field induced by the above mentioned body force field is given by

$$u_i = \int\int_{E'} (d_s + d_{s\alpha}x_\alpha')(1 - x_1'^2/a_1^2 - x_2'^2/a_2^2)^{-1/2}v_{si}(x_1 - x_1', x_2 - x_2', x_3)\,dx_1'\,dx_2', \quad (3.4)$$

where E' denotes the domain of the ellipse $1 - x_1'^2/a_1^2 - x_2'^2/a_2^2 > 0$. We must proceed to show that this quantity can be reconciled with the boundary conditions on the disc. An apparent obstacle to further progress is the fact that the "Green's functions" v_{si} are not known in closed form for a generally anisotropic elastic medium; among the conventional elastic symmetries the transversely isotropic medium appears to be the *ne plus ultra* in this regard. However, it is possible to proceed in a fashion analogous to Willis [5]. Defining the Fourier Transform by

$$\overline{f(\xi_1, \xi_2, \xi_3)} = (2\pi)^{-3/2}\int\int\int_{-\infty}^{\infty} f(x_1, x_2, x_3)e^{i\xi_j x_j}\,dx_1\,dx_2\,dx_3, \quad (3.5)$$

and transforming (3.1), we obtain

$$L_{ij}\bar{v}_{sj} = \delta_{si}(2\pi)^{-3/2}, \quad (3.6)$$

where

$$L_{ik} = c_{ijkl}\xi_l\xi_j = c_{klij}\xi_j\xi_l = L_{ki}. \quad (3.7)$$

Using (3.6) and inverting the transforms, we obtain

$$v_{si} = (8\pi^3)^{-1}\int\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} \mathcal{L}_{si}/L(\xi_1, \xi_2, \xi_3)e^{-i\xi_3x_3}\,d\xi_3\right\}e^{-i\xi_\alpha x_\alpha}\,d\xi_1\,d\xi_2. \quad (3.8)$$

In the above, $\mathcal{L}_{si}$ means the cofactor of L_{si}, L means the determinant $|L_{ij}|$, and the bracketed (ξ_1, ξ_2, ξ_3) merely indicates dependence on these variables.

The integration with respect to ξ_3 can be performed using Cauchy's theorem. The quantity L_{si}/L is homogeneous of degree -2 in (ξ_1, ξ_2, ξ_3) and therefore tends to zero as $|\xi_3| \to \infty$. Therefore, for $x_3 > 0$ we may close the contour in the lower half of the complex ξ_3 plane using Jordan's lemma to obtain

$$\int_{-\infty}^{\infty} \frac{\mathcal{L}_{si}}{L}e^{-i\xi_3x_3}\,d\xi_3 = -2\pi i\sum_{N=1}^{3}\frac{\mathcal{L}_{si}}{\partial L/\partial\xi_3}\{\xi_1, \xi_2, \xi_3^N(\xi_1, \xi_2)\}\,e^{-i\xi_3^N x_3}, \quad (3.9)$$

where $\xi_3 = \xi_3^N(\xi_1, \xi_2)$ are the roots of the equation

$$L(\xi_1, \xi_2, \xi_3) = 0 \quad (3.10)$$

with negative imaginary parts. The six roots of this equation cannot be real and therefore occur in complex conjugate pairs, this being a consequence of the ellipticity of the differential operator in (2.4). For $x_3 < 0$, the contour is closed in the upper half of the complex plane, and accordingly $\bar{\xi}_3{}^N$ replaces $\xi_3{}^N$ on the right-hand side of (3.9) while the initial minus sign is dropped. (The usual notation for complex conjugate is used throughout). The case $x_3 = 0$ is obtained from either of the above by letting $x_3 \to 0$.

Introducing the notation

$$\boldsymbol{\xi}^N = \{\xi_1, \xi_2, \xi_3{}^N(\xi_1, \xi_2)\}, \quad \bar{\boldsymbol{\xi}}^N = \{\xi_1, \xi_2, \bar{\xi}_3{}^N(\xi_1, \xi_2)\},$$
$$P_{mk}(\boldsymbol{\xi}^N) = -2\pi i \frac{\mathscr{L}_{mk}}{\partial L/\partial \xi_3}(\boldsymbol{\xi}^N), \tag{3.11}$$

and using the foregoing paragraph together with (3.8) gives

$$u_k = (8\pi^3)^{-1} \iint_{E'} (d_s + d_{s\alpha} x'_\alpha)(1 - x_1'^2/a_1^2 - x_2'^2/a_2^2)^{-1/2}\, \mathrm{d}x'_1\, \mathrm{d}x'_2 \iint_{-\infty}^{\infty} \sum_{N=1}^{3} P_{sk}(\boldsymbol{\xi}^N) \mathrm{e}^{-i\xi_3{}^N x_3 - i\xi_\alpha(x_\alpha - x'_\alpha)}\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2$$
$$\text{for} \quad x_3 \geq 0. \tag{3.12}$$

A somewhat similar representation may be written for $x_3 \leq 0$; similarly, in many cases occurring subsequently, representations for displacements and associated quantities are recorded for $x_3 > 0$ only, in the interests of brevity.

To make further progress, we appeal to a result proved by Willis[5] in connection with the problem of indentation of a generally anisotropic half-space by a smooth punch.

Let $f(\xi_1, \xi_2)$ and $m(\xi_1, \xi_2)$ have the properties

$$f(t\xi_1, t\xi_2) = t^{-1} f(\xi_1, \xi_2), \quad m(t\xi_1, t\xi_2) = t\, m(\xi_1, \xi_2), \quad t > 0 \tag{3.13}$$

and let p and q be positive integers or zero such that $p \geq q$, then

$$(2\pi)^{-1} \iint_{E'} (x'_1/a_1 + i x'_2/a_2)^p (x'_1/a_1 - i x'_2/a_2)^q (1 - x_1'^2/a_1^2 - x_2'^2/a_2^2)^{-1/2}\, \mathrm{d}x'_1\, \mathrm{d}x'_2$$
$$\iint_{-\infty}^{\infty} f(\xi_1, \xi_2) \mathrm{e}^{i\{m(\xi_1, \xi_2)x_3 - \xi_\alpha(x_\alpha - x'_\alpha)\}}\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2$$
$$= i \int_0^{2\pi} f(\eta_1/a_1, \eta_2/a_2)(\eta_1 + i\eta_2)^{p-q} J_{pq}\, \mathrm{d}\phi \tag{3.14}$$

and

$$\eta_1 = \cos\phi, \quad \eta_2 = \sin\phi,$$

and

$$J_{pq} = \int_0^1 y'^{2q+1}(1 - y'^2)^{-1/2}[-g + \sqrt{(g^2 - y'^2)}]^{p-q} / \sqrt{(g^2 - y'^2)}\, \mathrm{d}y' \tag{3.15}$$

while

$$g = m x_3 - \eta_1 x_1/a_1 - \eta_2 x_2/a_2.$$

Further

$$J_{00} = 1/2 \log \{(g+1)/(g-1)\}$$
$$J_{10} = 1 - 1/2g \log \{(g+1)/(g-1)\}. \tag{3.16}$$

It is readily verified that ξ_3^N and $P_{mk}(\boldsymbol{\xi}^N)$ have the properties stipulated for $m(\xi_1, \xi_2)$ and $f(\xi_1, \xi_2)$ respectively. Using the foregoing observations in conjunction with Willis' results (3.13)–(3.16), we can show that the displacement reduces to a first degree polynomial in x_i on the disc, thus solving the problem in principle. It proves convenient to make the detailed examination of the problem of translation and rotation separately.

4. TRANSLATION OF THE DISC

For the moment we take $dsa = 0$; this amounts to examining the translation of the disc i.e. putting $\omega_j = 0$.

In view of the last paragraph of Section 3, (3.12) yields

$$u_k = i(8\pi^2)^{-1} d_m \int_0^{2\pi} \sum_{N=1}^{3} P_{mk}\{\boldsymbol{\xi}^N(\eta_1/a_1, \eta_2/a_2)\} \log \{(g_N+1)/(g_N-1)\}\, \mathrm{d}\phi \quad \text{for} \quad x_3 \geq 0 \tag{4.1}$$

where

$$g_N = -\xi_3^N(\eta_1/a_1, \eta_2/a_2)x_3 - \eta_1 x_1/a_1 - \eta_2 x_2/a_2 \tag{4.2}$$

It is shown in Appendix A that

$$\sum_{N=1}^{3} P_{mk}\{\boldsymbol{\xi}^N(\xi_1, \xi_2)\} = \sum_{N=1}^{3} P_{mk}\{\boldsymbol{\xi}^N(-\xi_1, -\xi_2)\}. \tag{4.3}$$

Now

$$\begin{aligned} &\log \{(1 - \eta_1 x_1/a_1 - \eta_2 x_2/a_2)/(-1 - \eta_1 x_1/a_1 - \eta_2 x_2/a_2)\} \\ &= \log |(1 - \eta_1 x_1/a_1 - \eta_2 x_2/a_2)/(1 + \eta_1 x_1/a_1 + \eta_2 x_2/a_2)| - i\pi \end{aligned} \tag{4.4}$$

for points (x_1, x_2) on the disc, the first term on the right-hand side being odd in (η_1, η_2) for such points. Using this, the eveness in (ξ_1, ξ_2) of the first term in (4.3) together with (4.1), we see that the boundary condition (2.5) with $\omega_j = 0$ is satisfied if

$$c_k^m d_m = \alpha_k \tag{4.5}$$

where

$$c_k^m = (8\pi)^{-1} \int_0^{2\pi} \sum_{N=1}^{3} P_{mk}\{\boldsymbol{\xi}^N(\eta_1/a_1, \eta_2/a_2)\}\, \mathrm{d}\phi. \tag{4.6}$$

The system of eqns (4.5) is determinate, the matrix c_k^m being positive definite; see Appendix B.

The problem of translation of the disc is now solved in principle, the displacement field being given by (4.1) in terms of quite tractable integrals. Further, the resultant forces necessary to accomplish the specified translations are given by

$$P_i = 2\pi a_1 a_2 d_i \tag{4.7}$$

on using (3.3), and their evaluation in particular cases is a matter of calculating the integrals (4.6); this can always be done, by numerical methods if necessary.

It is evident from (4.5)–(4.7) that translation of the disc in any one co-ordinate direction in a generally anisotropic material (i.e. one which does not exhibit any elastic symmetry) requires resultant forces in all three such directions. Suppose, however, that there are planes of elastic symmetry perpendicular to the x_3 axis, then

$$c_{\alpha\beta 3\delta} = c_{\alpha 333} = 0, \quad (\alpha, \beta, \delta = 1, 2)$$

and it follows that

$$c_3{}^1 = c_3{}^2 = 0.$$

If $\alpha_1 = \alpha_2 = 0$, $\alpha_3 \neq 0$—a translation in the x_3 direction only—it follows that $d_1 = d_2 = 0$. Hence the resultant force which must be applied to the disc is in the x_3 direction also. Similar conclusions may be drawn if there are planes of elastic symmetry perpendicular to the other co-ordinate directions.

We now investigate the stress distribution on either side of the disc. Let q_+, q_- denote the values of the quantity q on the positive and negative sides of the disc $x_3 = 0\pm$. It is readily verified that

$$\partial u_k/\partial x_\beta)_\pm = 0 \tag{4.8}$$

by differentiating (2.5), or otherwise. Further, it follows from (4.1) that

$$(\partial u_k/\partial x_3)_+ = id_m(4\pi^2)^{-1}\int_0^{2\pi}\sum_{N=1}^{3} P_{mk}\{\xi^N(\eta_1/a_1, \eta_2/a_2)\}\xi_3{}^N(\eta_1/a_1, \eta_2/a_2)\{(\eta\cdot y)^2 - 1\}^{-1}\,d\phi, \tag{4.9}$$

where $\eta = (\eta_1, \eta_2)$ and $y = (x_1/a_1, x_2/a_2)$; $\partial u_k/\partial x_3)_-$ has the same form with opposite sign while $\bar{\xi}^N$ replaces ξ^N. Writing

$$\begin{aligned} i(2\pi)^{-1}\sum_{N=1}^{3} P_{mk}\{\xi^N(\eta_1/a_1, \eta_2/a_2)\}\xi_3{}^N(\eta_1/a_1, \eta_2/a_2) \\ = U_{mk}(\eta_1/a_1, \eta_2/a_2) + iV_{mk}(\eta_1/a_1, \eta_2/a_2) \end{aligned} \tag{4.10}$$

it follows from Appendix C that

$$V_{mk}(\eta_1/a_1, \eta_2/a_2) = -V_{mk}(-\eta_1/a_1, -\eta_2/a_2), \quad U_{mk} = b_{3mk3}|c_{3nl3}|^{-1}/2 \tag{4.11}$$

where $|c_{3mk3}|$ denotes the determinant of the matrix with elements c_{3mk3} and b_{3mk3} denote the corresponding cofactors. Since it is readily established that

$$\int_0^{2\pi}\{1-(y\cdot\eta)^2\}^{-1}\,d\phi = 2\pi/(1-y^2)^{1/2} \tag{4.12}$$

for points on the disc, it follows from (4.9)–(4.11) and (4.3) that

$$\partial u_k/\partial x_3)_+ = -d_m b_{3mk3}/\{2|c_{3nl3}|(1 - x_1{}^2/a_1{}^2 - x_2{}^2/a_2{}^2)^{1/2}\}, \tag{4.13}$$

the corresponding quantity with negative suffix being of the same form with opposite sign. In view of (2.2), (4.8), (4.13) it follows that

$$\begin{aligned} \tau_{3i})_\pm &= \mp d_m c_{3ik3} b_{3mk3}/\{|c_{3nl3}|(1 - x_1{}^2/a_1{}^2 - x_2{}^2/a_2{}^2)^{1/2}\} \\ &= \mp d_i/\{2(1 - x_1{}^2/a_1{}^2 - x_2{}^2/a_2{}^2)^{1/2}\} \\ &= \mp P_i/\{4\pi a_1 a_2(1 - x_1{}^2/a_1{}^2 - x_2{}^2/a_2{}^2)^{1/2}\} \end{aligned} \tag{4.14}$$

where we have used the result

$$c_{3ik3}b_{3mk3} = \delta_{im}|c_{3nl3}| \tag{4.15}$$

together with (4.7).

The result (4.14) is remarkable, showing that stress components on either side of the disc are independent of the elastic constants when the disc undergoes translational movement due to resultant forces P_i. An analogous result has been noted by Willis [5] in connection with the smooth indentation of an anisotropic elastic half space by a rigid punch. The ellipse is almost certainly atypical in this respect. The following is yet another result of a similar nature, apparently not alluded to in standard treatises on electricity, but derivable by elementary linear transformations: a conducting elliptic disc with semi-axes a_1, a_2 embedded in a generally anisotropic dielectric has charge density

$$Q/\{2\pi a_1 a_2(1-x_1^2/a_1^2-x_2^2/a_2^2)^{1/2}\}$$

when charged to amount Q, the electric field at infinity being zero. The derivation of this result by elementary linear transformations depends crucially on the property that an ellipse transforms into another ellipse. This suggests very strongly that in regard to the "constant-independence", both in the electrostatic and elastostatic examples, the ellipse is atypical.†

5. ROTATION OF DISC

We now examine the rotation of the disc, putting $d_s = 0$ in (3.3); this amounts to putting $\alpha_s = 0$. Rearranging (3.12) we obtain

$$u_k = (8\pi^3)^{-1}\int\int_{E'} d_{m\beta}x'_\beta(1-x_1'^2/a_1^2-x_2'^2/a_2^2)^{-1/2}\,\mathrm{d}x'_1\,\mathrm{d}x'_2$$

$$\int\int_{-\infty}^{\infty}\sum_{N=1}^{3}P_{mk}(\boldsymbol{\xi}^N)\mathrm{e}^{-i\{\xi_3^N(\xi_1,\xi_2)x_3+\xi_\alpha(x_\alpha-x'_\alpha)\}}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \quad \text{for} \quad x_3\geq 0. \tag{5.1}$$

A similar expression holds for $x_3<0$ with negative sign and $\bar{\xi}_3^N$ replacing ξ_3^N.

Recalling the condition (3.13) and using (3.14)–(3.17) we obtain

$$u_k = -i(8\pi^2)^{-1}\sum_{\alpha=1}^{2}d_{m\alpha}a_\alpha\int_0^{2\pi}\sum_{N=1}^{3}P_{mk}\{\boldsymbol{\xi}^N(\eta_1/a_1,\eta_2/a_2)\}$$
$$\cdot\, g_N\log\{(g_N+1)/(g_N-1)\}\eta_\alpha\,\mathrm{d}\phi \quad \text{for} \quad x_3\geq 0, \tag{5.2}$$

with a similar result for $x_3<0$ with negative sign and $\bar{\boldsymbol{\xi}}^N$ replacing $\boldsymbol{\xi}^N$. Hence for points on the disc

$$u_k = (8\pi)^{-1}\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}d_{m\alpha}a_\alpha(x_\beta/a_\beta)\int_0^{2\pi}\sum_{N=1}^{3}P_{mk}\{\boldsymbol{\xi}^N(\eta_1/a_1,\eta_2/a_2)\}\eta_\alpha\eta_\beta\,\mathrm{d}\phi. \tag{5.3}$$

Adopting the notation

$$K_{ij}=K_{ji}=\begin{cases}(8\pi)^{-1}\displaystyle\int_0^{2\pi}\sum_{N=1}^{3}P_{ji}\{\boldsymbol{\xi}^N(\eta_1/a_1,\eta_2/a_2)\}\eta_1^2\,\mathrm{d}\phi, & 1\leq i\leq 3,\quad 1\leq j\leq 3\\ (8\pi)^{-1}\displaystyle\int_0^{2\pi}\sum_{N=1}^{3}P_{j-3\ i}\{\boldsymbol{\xi}^N(\eta_1/a_1,\eta_2/a_2)\}\eta_1\eta_2\,\mathrm{d}\phi, & 1\leq i\leq 3,\quad 4\leq j\leq 6\\ (8\pi)^{-1}\displaystyle\int_0^{2\pi}\sum_{N=1}^{3}P_{j-3\ i-3}\{\boldsymbol{\xi}^N(\eta_1/a_1,\eta_2/a_2)\}\eta_2^2\,\mathrm{d}\phi, & 4\leq i\leq 6,\quad 4\leq j\leq 6\end{cases} \tag{5.4}$$

and letting $\mathbf{f}$ denote the column vector with elements

$$a_1d_{11},\ a_1d_{21},\ a_1d_{31},\ a_2d_{12},\ a_2d_{22},\ a_2d_{32},$$

and $\boldsymbol{\Omega}$ that with elements

†One cannot use linear transformations to derive the above results (for a generally anisotropic elastic medium) from those of an isotropic medium unless nine somewhat artificial relations exist between the elastic constants [7].

$$0, a_1\omega_3, -a_1\omega_2, -a_2\omega_3, 0, a_2\omega_1$$

we find that the boundary conditions for rotation, (2.5) with $a_i = 0$, are satisfied provided

$$\mathbf{Kf} = \mathbf{\Omega}$$

where we have used (5.3). This is a determinate system as the 6×6 matrix $\mathbf{K}$ is non-singular, see Appendix B.

Thus the problem of rotation of the disc is solved in principle, the displacement field being given by (5.1) in terms of quite tractable integrals. The torques about the origin necessary to accomplish the specified rotation are given by

$$G_1 = 2\pi a_1 a_2^3 d_{32}/3, \quad G_2 = -2\pi a_1^3 a_2 d_{31}/3, \quad G_3 = 2\pi(a_1^3 a_2 d_{21} - a_1 a_2^3 d_{12})/3. \tag{5.6}$$

Their evaluation in particular cases is a matter of calculating the integrals (5.4); this can always be done by numerical methods if necessary.

We now investigate some of the consequences due to the existence of planes of elastic symmetry.

$0x_1x_2$ *a plane of symmetry.* Here

$$K_{13} = K_{23} = K_{34} = K_{35} = K_{16} = K_{26} = K_{46} = K_{56} = 0$$

and eqn (5.5) separates into two independent systems in (d_{31}, d_{32}) and $(d_{11}, d_{12}, d_{21}, d_{22})$. If $\omega_1 = \omega_2 = 0$ then $d_{31} = d_{32} = 0$ and hence no torques G_1, G_2 are required. If $\omega_3 = 0$, all the $d_{s\alpha}$ other than d_{31}, d_{32} are zero, and no G_3 is required.

$0x_1x_3$ *a plane of symmetry.* Here

$$K_{12} = K_{14} = K_{16} = K_{23} = K_{25} = K_{34} = K_{36} = K_{45} = K_{56} = 0$$

and (5.5) uncouples into two independent systems in (d_{11}, d_{31}, d_{22}) and (d_{21}, d_{12}, d_{32}). If $\omega_2 = 0$ then $d_{11} = d_{31} = d_{22} = 0$ and in view of $(5.6)_2$, no torque component G_2 is required. If $\omega_1 = \omega_3 = 0$ then $d_{21} = d_{12} = d_{32} = 0$ and no torque component G_3 is required.

$0x_2x_3$ *a plane of symmetry.* This case is clearly analogous to the last one and details are, therefore, superfluous.

Orthotropic symmetry. Suppose there is symmetry with respect to all three co-ordinate planes (symmetry with respect to any two implies symmetry with respect to the third). In this case $d_{11} = d_{12} = 0$. If $\omega_1 \neq 0, \omega_2 = \omega_3 = 0$ then $d_{32} \neq 0$ and all the others are zero. If $\omega_2 \neq 0, \omega_1 = \omega_3 = 0$ then $d_{31} \neq 0$ and all the others are zero. Hence a rotation of the disc about the x_1 axis alone can be accomplished by means of a torque G_1 alone, and a rotation about the x_2 axis can be accomplished by means of a torque G_2 alone. In fact,

$$G_1 = 2\pi a_1 a_2^3 \omega_1/3K_{66}, \quad G_2 = 2\pi a_1^3 a_2 \omega_2/3K_{33}. \tag{5.7}$$

In the case of $\omega_3 \neq 0$, $\omega_1 = \omega_2 = 0$, d_{21}, d_{12} and hence G_3 are calculated from

$$\begin{aligned} K_{22}a_1 d_{21} + K_{24}a_2 d_{12} &= a_1\omega_3 \\ K_{42}a_1 d_{21} + K_{44}a_2 d_{12} &= -a_2\omega_3. \end{aligned} \tag{5.8}$$

We now investigate the stress distribution on either side of the disc, the suffixes + and − having the same significance as in the last section. Differentiating $(2.5)_1$ with respect to x_β gives

$$\partial u_k/\partial x_\beta)_\pm = e_{kl\beta}\omega_l \tag{5.9}$$

the derivative being continuous across the disc. This result can also be obtained by differentiating

(5.2). Differentiating (5.2) with respect to x_3 and letting $x_3 \to 0+$ on the disc yields

$$\partial u_k/\partial x_3)_+ = i(4\pi^2)^{-1} \sum_{\alpha=1}^{2} d_{m\alpha}a_\alpha \int_0^{2\pi} \sum_{N=1}^{3} P_{mk}\{\xi^N(\eta_1/a_1, \eta_2/a_2)\}\xi_3^N(\eta_1/a_1, \eta_2/a_2)$$
$$\times [\eta \cdot y\{(\eta \cdot y)^2 - 1\}^{-1} + (1/2)\log\{(1-\eta \cdot y)/(-1-\eta \cdot y)\}]\eta_\alpha \, d\phi. \quad (5.10)$$

Recalling (4.4), (4.10) and (4.11), (5.10) reduces to

$$\partial u_k/\partial x_3)_+ = (4\pi)^{-1} \sum_{\alpha=1}^{2} d_{m\alpha}a_\alpha b_{3mk3}|c_{3nl3}|^{-1} \int_0^{2\pi} [(\eta \cdot y)\{(\eta \cdot y)^2 - 1\}^{-1}$$
$$+ (1/2)\log|(1-\eta \cdot y)/(1+\eta \cdot y)|]\eta_\alpha \, d\phi + (1/4)\sum_{\alpha=1}^{2} d_{m\alpha}a_\alpha \int_0^{2\pi} V_{mk}(\eta_1/a_1, \eta_2/a_2)\eta_\alpha \, d\phi. \quad (5.11)$$

A similar analysis shows that the corresponding quantity with negative suffix has the same form but the first term of (5.11) has negative sign. It is readily established that

$$\int_0^{2\pi} [(\eta \cdot y)\{(\eta \cdot y)^2 - 1\}^{-1} + (1/2)\log|(1-\eta \cdot y)/(1+\eta \cdot y)|]\eta_\alpha \, d\phi$$
$$= -2\pi y_\alpha(1-y^2)^{-1/2} \quad (5.12)$$

Using this in connection with (5.11) et seq. yields

$$\partial u_k/\partial x_3)_\pm = \mp d_{m\alpha}b_{3mk3}/(2|c_{3nl3}|) \cdot [1 - x_1^2/a_1^2 - x_2^2/a_2^2)^{-1/2}$$
$$+ (1/4)\sum_{\alpha=1}^{2} d_{m\alpha}a_\alpha \int_0^{2\pi} V_{mk}(\eta_1/a_1, \eta_2/a_2)\eta_\alpha \, d\phi. \quad (5.13)$$

Using (2.2), (4.16), (5.9), (5.13) together with the definition of **f** we obtain (for points on either side of the disc $x_3 = 0\pm$)

$$\tau_{3i})_\pm = \mp d_{i\alpha}x_\alpha/\{2(1 - x_1^2/a_1^2 - x_2^2/a_2^2)^{1/2}\}$$
$$+ (1/4)c_{3ik3}(f_m I^1_{mk} + f_{m+3}I^2_{mk}) + c_{3ik\beta}e_{kl\beta}\omega_l \quad (5.14)$$

where

$$I^\alpha_{mk} = \int_0^{2\pi} V_{mk}(\eta_1/a_1, \eta_2/a_2)\eta_\alpha \, d\phi.$$

We now investigate the way in which (5.14) reduces if one or more of the co-ordinate axes are perpendicular to planes of elastic symmetry. Suppose that the Ox_1x_2 plane is one of elastic symmetry. Recalling (3.7) we see that

$$L_{ik}(\xi_1, \xi_2, \xi_3) = L_{ik}(\xi_1, \xi_2, -\xi_3) = L_{ik}(-\xi_1, -\xi_2, \xi_3).$$

Hence, in view of (4.10), we have

$$V_{mk}(\eta_1/a_1, \eta_2/a_2) = V_{mk}(-\eta_1/a_1, -\eta_2/a_2)$$

and, in view of (4.11), it follows that

$$I^\alpha_{mk} = 0. \quad (5.15)$$

Suppose $\omega_3 = 0$ in addition, then d_{31} and d_{32} are the only non vanishing $d_{s\alpha}$, and recalling $(5.6)_{1,2}$, we have

$$\tau_{33})_\pm = \mp 3(G_1x_2/a_1a_2^3 - G_2x_1/a_2a_1^3)/\{4\pi(1-x_1^2/a_1^2-x_2^2/a_2^2)^{1/2}\}$$
$$\tau_{3\alpha})_\pm = c_{3\alpha 3\beta}e_{3\gamma\beta}\omega_\gamma. \tag{5.16}$$

The result $(5.16)_1$ is remarkable, being independent of the elastic constants; as in the case of (4.14), it appears likely that the ellipse is atypical in this respect.

Suppose $\omega_1 = \omega_2 = 0$ (in addition to Ox_1x_2 being a plane of elastic symmetry), then $d_{31} = d_{32} = 0$ and

$$\tau_{33})_\pm = 0, \quad \tau_{3\gamma})_\pm = \mp d_{\gamma\alpha}x_\alpha/\{2(1-x_1^2/a_1^2-x_2^2/a_2^2)^{1/2}\}. \tag{5.17}$$

Suppose now that the ellipse degenerates to a circle ($a_1 = a_2 = a$, say), and suppose that the elastic material is tranversely isotropic with respect to the Ox_3 axis, it follows from (5.4), (5.6), (5.8) and (5.17) that the polar (r, θ) components of the shear stress on either side of the circular disc are given by

$$\tau_{\theta 3})_\pm = \mp 3G_3r/\{4\pi a^2(a^2-r^2)^{1/2}\}, \quad \tau_{r3})_\pm = 0.$$

These expressions are also independent of the elastic constants.

Finally we remark that if $0x_2x_3$ is a plane of elastic symmetry it follows that $I^1_{mk} = 0$; also, if $0x_1x_3$ is a plane of elastic symmetry it follows that $I^2_{mk} = 0$.

6. DISC PERTURBING UNIFORM STRESS FIELD

The problem of a rigid elliptic disc perturbing a uniform stress field deserves mention as it is amenable to the same techniques used for the problem of rotation. The problem is to solve (2.4) subject to

$$u_i = E_{ij}x_j, \quad x_ix_i \to \infty,$$

and

$$u_i = 0 \quad \text{on} \quad \{x_i: 1-x_1^2/a_1^2-x_2^2/a_2^2>0, \quad x_3=0\}, \tag{6.1}$$

where E_{ij} are constants.

Writing

$$u_i = E_{ij}x_j + u'_i, \tag{6.2}$$

We have

$$c_{ijkl}\partial^2u'_k/\partial x_l\partial x_j = 0, \tag{6.3}$$

subject to

$$u'_i = -E_{ij}x_j \quad \text{on} \quad \{x_i: 1-x_1^2/a_1^2-x_2^2/a_2^2>0, \quad x_3=0\},$$
$$u'_i \to 0, \quad x_ix_i\to\infty. \tag{6.4}$$

Dropping the primes from u'_i it is evident that (5.1) provides a suitable representation for u_i, the boundary condition (6.4) being satisfied on choosing $d_{s\alpha}$ from

$$\mathbf{Kf} = -\mathbf{E}, \tag{6.5}$$

where $\mathbf{E}$ denotes the column vector with elements

$$a_1E_{11}, \quad a_1E_{21}, \quad a_1E_{31}, \quad a_2E_{12}, \quad a_2E_{22}, \quad a_2E_{32}.$$

The stresses on either side of the disc $x_3 = O\pm$ can be found in a manner similar to (5.14). It is found that

$$\tau_{3i} = \mp d_{i\alpha}x_\alpha/\{2(1-x_1^2/a_1^2-x_2^2/a_2^2)^{1/2}\}+(1/4)c_{3ik3}(f_mI^1_{mk}+f_{m+3}I^2_{mk})-c_{3ik\beta}E_{k\beta}. \quad (6.6)$$

As in the case of rotation, the existence of planes of elastic symmetry coinciding with the co-ordinate planes reduces the above to simpler form.

The result (3.14) can also be made to yield the distrubed elastic field due to an embedded rigid elliptic disc in an infinite medium when the stress at infinity is a polynomial in x_1, x_2, x_3 but this is not pursued here.

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APPENDIX A

Consider $\int_\Gamma(\mathscr{L}_{mk}/L)(\xi_1,\xi_2,\xi_3)\,d\xi_3$ where Γ is a simple closed contour in the complex ξ_3 plane enclosing all the zeros of $L(\xi_3)=0$; as Γ expands to infinity it is apparent that the integral tends to zero and so the sum of the residues of the integrand vanishes (Cauchy's Theorem) i.e.

$$\sum_{N=1}^{3}[P_{mk}\{\xi_1,\xi_2,\xi_3^N(\xi_1,\xi_2)\}+P_{mk}\{\xi_1,\xi_2,\bar{\xi}_3^N(\xi_1,\xi_2)\}]=0, \quad (A1)$$

It is clear that if $\xi_3^N(\xi_1,\xi_2)$ is a root of $L(\xi_3)=0$, so is $-\xi_3^N(-\xi_1,-\xi_2)$. Since $\xi_3^N(-\xi_1,-\xi_2)$ has a negative imaginary part by definition, $-\xi_3^N(-\xi_1,-\xi_2)$ has a positive imaginary part, and

$$\xi_3^N(-\xi_1,-\xi_2)=-\bar{\xi}_3^M(\xi_1,\xi_2), \quad (A2)$$

for some M not necessarily equal to N, as the roots of $L(\xi_3)=0$ occur in complex conjugate pairs. Also

$$P_{mk}\{\xi_1,\xi_2,\xi_3^N(\xi_1,\xi_2)\}=-P_{mk}\{-\xi_1,-\xi_2,-\xi_3^N(\xi_1,\xi_2)\} \quad (A3)$$

since $P_{mk}(\xi_1,\xi_2,\xi_3)$ is a homogeneous rational function of degree -1 in (ξ_1,ξ_2,ξ_3). It follows from (A1)–(A3) that

$$\sum_{N=1}^{3}P_{mk}\{\xi_1,\xi_2,\xi_3^N(\xi_1,\xi_2)\}=\sum_{N=1}^{3}P_{mk}\{-\xi_1,-\xi_2,\xi_3^N(-\xi_1,-\xi_2)\}. \quad (A4)$$

APPENDIX B

Since the energy stored in a deformed elastic material is positive definite, it follows that

$$G_i\omega_i+F_i\alpha_i\geqslant 0,$$

the equality sign occurring if and only if $\alpha_i=\omega_i=0$. In the case of translation alone ($\omega_i=0$), it follows that

$$d_ic_i^jd_j\geqslant 0,$$

on remembering $F_i=2\pi a_1a_2d_i$; the equality sign holds if and only if $d_i=0$. Thus the matrix c_i^j is positive definite and, therefore, non-singular. Similarly, in the case of rotation alone ($\alpha_i=0$), it follows that

$$f_iK_{ij}f_j\geqslant 0,$$

the equality sign holding if and only if $f_i=0$. Thus, the 6×6 matrix K_{ij} is positive definite, and, therefore, non-singular.

APPENDIX C

Consider $\int_\Gamma(\mathscr{L}_{mk}/L)(\xi_1,\xi_2,\xi_3)\,d\xi_3$ where Γ is a simple closed contour in the complex ξ_3 plane enclosing all the zeros of $L(\xi_3)=0$. Now the integrand is $O(\xi_3^{-1})$ as $|\xi_3|\to\infty$, and the residue at infinity is given by

$$-\lim_{\xi_3\to\infty}\frac{\mathscr{L}_{mk}}{L}\xi_3^2,$$

which has the form

$$-\lim_{\xi_3\to\infty}\xi_3^2\Big(\sum_{r+s+t=4}c_{rst}\xi_1^r\xi_2^s\xi_3^t\Big)\Big/\Big(\sum_{r_1+s_1+t_1=6}d_{r_1s_1t_1}\xi_1^{r_1}\xi_2^{s_1}\xi_3^{t_1}\Big).$$

An inspection of the latter shows that the dominant term above corresponds to the coefficient of ξ_3^4 in $\mathscr{L}_{mk}$, and the dominant term below corresponds to the coefficient of ξ_3^6 in L; the latter term is given by c_{3nl3}—the determinant with elements c_{3nl3}—while the former term is given by b_{3nl3}—the cofactor corresponding to the element c_{3nl3}. In view of the foregoing, Cauchy's Theorem yields

$$\sum_{N=1}^{3}\left[\frac{\mathscr{L}_{mk}}{\partial L/\partial \xi_3}\{\xi_1,\xi_2,\xi_3^N(\xi_1,\xi_2)\}\xi_3^N(\xi_1,\xi_2)+\frac{\mathscr{L}_{mk}}{\partial L/\partial \xi_3}\{\xi_1,\xi_2,\bar{\xi}_3^N(\xi_1,\xi_2)\}\bar{\xi}_3^N(\xi_1,\xi_2)\right]=b_{3mk3}|c_{3nl3}|^{-1}. \tag{C1}$$

Arguments similar to those employed in Appendix A (post (A1)) yield

$$\sum_{N=1}^{3}\left[\frac{\mathscr{L}_{mk}}{\partial L/\partial \xi_3}\{\xi_1,\xi_2,\xi_3^N(\xi_1,\xi_2)\}\right]\xi_3^N(\xi_1,\xi_2)=\sum_{N=1}^{3}\left[\frac{\mathscr{L}_{mk}}{\partial L/\partial \xi_3}\{-\xi_1,-\xi_2,\xi_3^N(-\xi_1,-\xi_2)\}\right]\xi_3^N(-\xi_1,-\xi_2) \tag{C2}$$

Writing

$$\sum_{N=1}^{3}\frac{\mathscr{L}_{mk}}{\partial L/\partial \xi_3}\{\xi_1,\xi_2,\xi_3^N(\xi_1,\xi_2)=U_{mk}(\xi_1,\xi_2)+iV_{mk}(\xi_1,\xi_2),$$

(C1) and (C2) yield

$$U_{mk}(\xi_1,\xi_2)=U_{mk}(-\xi_1,-\xi_2)=b_{3mk3}/(2|c_{3nl3}|), \tag{C3}$$

and

$$V_{mk}(\xi_1,\xi_2)=-V_{mk}(-\xi_1,-\xi_2). \tag{C4}$$